Packing and Cutting Problems

Enrico Malaguti

DEIS - University of Bologna

July, 2012

Outline

Knapsack Problems

Knapsack Problem - KP01

Given a set of m items, each item $i = 1, \ldots, m$ having a positive profit p_i and a positive weight w_i, and a knapsack of capacity C, select a subset of items of maximum profit without exceeding the knapsack capacity.

We use *binary* variables x_i, $i = 1, \ldots, m$, taking value 1 when item i is selected and 0 otherwise. A possible *ILP* model reads:

$$\max \sum_{i=1}^{m} p_i x_i$$

$$\sum_{i=1}^{m} w_i x_i \leq C$$

$$x_i \in \{0, 1\}, \ i = 1, \ldots, m$$

How would you solve the continuos relaxation of the $KP01$?

Bounded Knapsack Problem

Given a set of m item classes, each item class $i = 1, \ldots, m$ having a positive profit p_i, a positive weight w_i **and available in d_i copies**, and a knapsack of capacity C, select a subset of items of maximum profit without exceeding the knapsack capacity.

We use *integer* variables $x_i, \ i = 1, \ldots, m$, denoting the number of times item i is selected. A possible *ILP* model reads:

$$\max \sum_{i=1}^{m} p_i x_i$$

$$\sum_{i=1}^{m} w_i x_i \leq C$$

$$0 \leq x_i \leq d_i, \ integer, \ i = 1, \ldots, m$$

Bounded Knapsack Problem

Two ways of solving the BKP as a KP01:

- consider the equivalent KP01 problem where for each class i there are d_i separated identical items $i_1, \ldots, i_{d_i}$;
- consider an equivalent KP problem where for each class i there are $\lceil \log d_i \rceil$ separated items $i_l,\ l = 0, \ldots, \lceil \log d_i \rceil$ with $p_{i_l} = 2^l p_i$, $w_{i_l} = 2^l w_i$; and the additional constraints: $\sum_{l=0}^{\lceil \log d_i \rceil} 2^l x_{i_l} \leq d_i$.

Bin Packing and Cutting Stock Problems

Bin Packing Problem

Given a set of m items, each item $i = 1, \ldots, m$ having a positive weight w_i, and n identical bins of capacity C, pack all items in the minimum number of bins.
We use *binary* variables $x_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n$, taking value 1 when item i is inserted into bin j; and *binary* variables $y_j,\ j = 1, \ldots, n$, taking value 1 when bin j is used.

$$\text{M1-BPP} \quad \min \sum_{j=1}^{n} y_j$$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, \ldots, m$$

$$\sum_{i=1}^{m} w_i x_{ij} \leq C y_j,\ j = 1, \ldots, n$$

$$x_{ij}, y_j \in \{0, 1\},\ i = 1, \ldots, m;\ j = 1, \ldots, n$$

Bin Packing Problem

What is the value z of the optimal solution of the continuos relaxation of the $M1 - BPP$?
How would you solve the continuos relaxation of $M1 - BPP$?

$$z = \frac{\sum_{i=1}^{m} w_i}{C}$$

$$y_j = z/n, \; j = 1, \dots, n; \; x_{ij} = 1/n, \; i = 1, \dots, m; \; j = 1, \dots, n$$

$M1 - BPP$ is a *weak* model: it provides a trivial (and far from the optimal integer value) lower bound. In addition, the model is highly symmetric: given an integer solution to $M1 - BPP$ of value k, we can construct $\binom{n}{k} k!$ equivalent solutions.

Cutting Stock Problem

Given a set of m items classes, where for each $i = 1, \ldots, m$ there are d_i identical copies with positive weight w_i, and n bins of capacity C, pack all items in the minimum number of bins.

We use *integer* variables $x_{ij},\ i = 1, \ldots, m,\ j = 1, \ldots, n$, denoting the number of copies of items of class i inserted into bin j; and *binary* variables $y_j,\ j = 1, \ldots, n$, taking value 1 when bin j is used.

$$\min \sum_{j=1}^{n} y_j$$

$$\sum_{j=1}^{n} x_{ij} = d_i, \qquad i = 1, \ldots, m$$

$$\sum_{i=1}^{m} w_i x_{ij} \leq C y_j, \qquad j = 1, \ldots, n$$

$$x_{ij} \ \textit{integer},\ y_j \in \{0, 1\}, \quad i = 1, \ldots, m;\ j = 1, \ldots, n$$

Bin Packing Problem

Let consider the collection $\mathcal{S}'$ of subsets of items which can fit in one bin:

$$\mathcal{S}' = \{S \subseteq \{1, \ldots, m\} : \sum_{i \in S} w_i \leq C\}$$

A binary variable x_S is associated with each one of the exponentially many sets S, taking value one if the set is selected and 0 otherwise. The corresponding *Set Partitioning* model reads:

$$\text{SP-BPP} \quad \min \sum_{S \in \mathcal{S}'} x_S$$

$$\sum_{S \in \mathcal{S}' : i \in S} x_S = 1, \; i = 1, \ldots, m$$

$$x_S \in \{0, 1\}, \; S \in \mathcal{S}'$$

$SP - BPP$ has $O(2^n)$ variables (collections represented as binary vectors).

Bin Packing Problem

Let consider the collection $\mathcal{S}$ of *maximal* subsets of items which can fit in one bin:

$$\mathcal{S} = \{S \subseteq \{1, \ldots, m\} : \sum_{i \in S} w_i \leq C, \sum_{i \in S \cup \{j\}} w_i > C, \ \forall j \notin S\}$$

The *Set Covering* model for the *BPP* reads:

$$\text{SC-BPP} \quad \min \sum_{S \in \mathcal{S}} x_S$$

$$\sum_{S \in \mathcal{S} : i \in S} x_S \geq 1, \ i = 1, \ldots, m$$

$$x_S \in \{0, 1\}, \ S \in \mathcal{S}$$

Observation - Models $SP - BPP$ and $SC - BPP$ are *equivalent*.

Consider the following example of BPP:
$n = 5,\ w = [7, 4, 1, 4, 4],\ C = 10.$

How "strong" is model $SC - BPP$?

- For a given instance of BPP, Let $z(LP(SC - BPP))$ be the optimal value of the continuos relaxation of model $SC - BPP$, and $z^*(BPP)$ the optimal value of the BPP (integer solution). How far are these values in the *worst* case?

  Conjecture - It does not exist an instance of BPP where:
  $\lceil z(LP(SC - BPP)) \rceil + 1 < z^*(BPP)$.
- Model $SC - BPP$ is not symmetric with respect to the bin selection.

Cutting Stock Problem

Let consider the collection $\mathcal{S}$ of subsets of items which can fit in one bin: they can be denoted by m-dimensional integer vectors a^S such that

$$\sum_{i=1}^{m} a_i^S w_i \leq C$$

where a_i^S is the number of copies of i in subset S (0-1 for the BPP). The *"Set Covering"* model for the *CSP* reads:

$$\text{SC-CSP} \quad \min \sum_{S \in \mathcal{S}} x_S$$

$$\sum_{S \in \mathcal{S}} a_i^S x_S \geq d_i, \ i = 1, \ldots, m$$

$$x_S \ \textit{integer}, \ S \in \mathcal{S}$$

Cutting Stock Problem

The dual of the LP-relaxation of the $SC - CSP$ reads:

$$\text{D(SC-CSP)} \quad \max \sum_{i=1}^{m} d_i \pi_i$$

$$\sum_{i=1}^{m} a_i^S \pi_i \leq 1, \ S \in \mathcal{S}$$

$$\pi_i \geq 0, \ i = 1, \ldots, m$$

The $SC - CSP$ LP-relaxation is initialized with a subset of the columns (variables). Given an optimal solution x^*, π^* to the so-called *restricted master problem*, we do not have optimality if it exists a violated dual constraint (eq., a column with negative reduced cost) for some set $S^* \in \mathcal{S}$.

$$\sum_{i=1}^{m} a_i^{S^*} \pi_i > 1$$

Column Generation for the SC-BPP

The column generation problem (slave problem) can be solved by introducing binary variables z_i denoting the number of copies of item i is in S^*, and checking if the following system has a solution:

$$\sum_{i=1}^{m} \pi_i^* z_i > 1$$

$$\sum_{i=1}^{m} w_i z_i \leq C$$

$$0 \leq z_i \leq d_i, \ \ integer, \ \ i = 1, \ldots, m$$

If such a subset does not exist, (x^*, π^*) is optimal.

Column Generation for the SC-BPP

We can directly look for the column having the *largest negative reduced cost* by solving the following ILP:

$$\max \sum_{i=1}^{m} \pi_i^* z_i$$

$$\sum_{i=1}^{m} w_i z_i \leq C$$

$$0 \leq z_i \leq d_i, \ \textit{integer}, \ i = 1, \ldots, m$$

Which is a bounded KP (01KP for the BPP) with profits $\pi_i^*, \ i = 1, \ldots, m$.

Branch and Price for the Bin Packing Problem

If the optimal solution of the $CS - BPP$ is fractional, column generation is embedded in a Branch-and-Bound scheme.

- Choose two columns S and S' with $0 < x_S < 1$, $0 < x'_S < 1$
- Select two items i, j such that $a_i^S = a_j^S = 1$, $a_i^{S'} = 1$, $a_j^{S'} = 0$
- Consider the two subproblems where: 1) i and j are collapsed into a single item having weight $w_i + w_j$ and 2) i and j cannot stay in the same bin.

The resulting slave problems need to be modified as follows:

1. Is still a KP01 with one new item of weight $w_i + w_j$ replacing i and j.
2. Is a KP01 with and additional incompatibility constraint: $z_i + z_j \leq 1$.

Branch and Price for the Cutting Stock Problem

Idea: find an item i, a value $v \in \{0, \dots, d_i\}$ and a "sense" $\geq$ or $<$ such that $\sum_{S:a_i^s \geq v} x_S = \alpha$ is fractional (or, eq., consider $\sum_{S:a_i^s < v} x_S$). Branch by imposing:

1. $\sum_{S:a_i^s \geq v} x_S \leq \lfloor \alpha \rfloor$
2. $\sum_{S:a_i^s \geq v} x_S \geq \lceil \alpha \rceil$

In general we may need more than one item/level/sense (component bounds) to have α fractional. Let $f = \sum_S (x_S - \lfloor x_S \rfloor)$ and $\mathcal{S}(B)$ a set of columns satisfying a set of component bounds B.

Proposition (Vanderbeck, 2000). Given a fractional solution x, there exists a set of component bounds B with $|B| \leq \lceil \log f \rceil + 1$ such that $\sum_{S \in \mathcal{S}(B)} x_S$ is fractional.

We add the following valid inequality (with associated ψ variable) to each node of the Branch-and-Price:

$$\sum_{s \in S} x_s \geq LB_{node} \quad (1)$$

where LB_{node} is the lower bound at the node. We can omit component bounds B of kind "$<$" because they are implied as complements of "$\geq$".

Assume that one component bound suffice for branching. Let denote by $K = K1 \cup K2$ the set of branching constraints ($\leq \lfloor \alpha \rfloor$ and $\geq \lceil \alpha \rceil$), and $\mu_k, \; k \in K1$ and $\nu_k, \; k \in K2$ the associated dual variables.

Branch and Price for the Cutting Stock Problem

Let $s(k)$, $v(k)$ be the item and value associated with constraint k.

$$\min \sum_{S \in \mathcal{S}} x_S$$

$$\sum_{S \in \mathcal{S}} a_i^S x_S \geq d_i, \; i = 1, \ldots, m$$

$$\sum_{s \in S} x_S \geq LB_{node}$$

$$\sum_{S : a_i^S(k) \geq v(k)} x_S \leq \lfloor \alpha \rfloor, \; k \in K1 \qquad \sum_{S : a_i^S(k) \geq v(k)} x_S \geq \lceil \alpha \rceil, \; k \in K2$$

thus dual constraints read:

$$\sum_{i=1}^{m} \pi_i a_s^S + \psi - \sum_{k \in K^1 : a_s^S(k) \geq v(k)} \mu_k + \sum_{k \in K^2 : a_s^S(k) \geq v(k)} \nu_k \leq 1, \; S \in \mathcal{S}$$

Branch and Price for the Cutting Stock Problem

The slave problem is modified as follows:

$$\max \sum_{i=1}^{m} \pi^* z_i - \sum_{k \in K^1} g_k \mu_k^* + \sum_{k \in K^2} h_k \nu_k^* + \psi \tag{2}$$

$$(d_s(k) - v(k) + 1) g_k \geq (z_{s(k)} - v(k) + 1) \qquad k \in K^1 \tag{3}$$

$$v(k) h_k \leq z_{s(k)} \qquad k \in K^2 \tag{4}$$

$$[\text{bounded KP constraints}] \tag{5}$$

$$z_i \in \mathbb{Z}^+ \quad i = 1, \ldots, m \tag{6}$$

$$g_k \in \{0, 1\} \qquad k \in K^1 \tag{7}$$

$$h_k \in \{0, 1\} \qquad k \in K^2 \tag{8}$$

Similarly (but at cost of additional variables and constraints) we can tackle the case when more than one component bound is needed for branching.

Branch and Price for the Cutting Stock Problem

Alternative strategy: remember that we can map the items i with demand $d_i > 1$ to $\lceil \log d_i \rceil$ separated items i_l, $l = 0, \ldots, \lceil \log d_i \rceil$. Thus we keep columns S of original items in the master problem and columns $\bar{S}$ of "new" items in the slave: $a_i^S = \sum_{l=0}^{\lceil \log d_i \rceil} 2^l a_{i_l}^{\bar{S}}$.

- Select and item i_l such that $\sum_{S<->\bar{S}, i_l=1} x_s = \alpha$ is fractional.
- Branch by imposing integrality of α.

In the slave problem, we only have to modify the profit (dual variable) associated with item i_l.

2D Packing and Cutting

2D Packing and Cutting

Problem: given a set of 2D items of whatever shape, obtain the requested items by minimizing the use of stock material, or maximize the profit of items obtained from the available stock material.

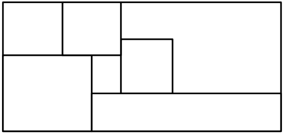
Common restrictions:

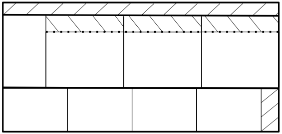
- Often it is assumed that items are rectangular and the stock is rectangularly shaped as well (bins or strips), in addition, items are obtained from the stock with vertical or horizontal cuts;
- general model: space is discretized and no overlapping of the items is imposed.

2D Packing and Cutting

Additional features may be considered depending on the way the items are obtained from the stock:

- guillotine cuts, i.e., cuts that are parallel to the sides of the stock and cross the stock from one side to the other;
- staged cutting, where each stage consists in a set of parallel guillotine cuts performed on the shape obtained in the previous stage.
- 2 staged guillotine cutting produces shelves, i.e., slices of the stock rectangle with the same width, and height coincident with the height of the tallest item cut off from it.
- items may or may not be rotated;
- bins or strips of different size may be available.





2dimensional 2staged Guillotine Knapsack Problem (2DKP)

We are given a unique rectangular stock with height H and width W, and a list of m rectangular shapes to be cut. Each shape's type i $(i = 1, \ldots, m)$ is characterized by a height h_i, a width w_i, a profit p_i, and an upper bound ub_i indicating the maximum number of items of type i which can be cut.
The problem calls for the determination of a 2staged Guillotine cutting pattern maximizing the sum of the profits of the cut items.

Let $\mathcal{S} := \{1, \ldots, NS\}$ be the set of all "feasible" shelves, satisfying the condition:

$$\sum_{i=1}^{m} w_i r_i^s \leq W$$

where r_i^s $(0 \leq r_i^s \leq ub_i,\ i = 1, \ldots, m)$ denotes the number of items with shape's type i which are cut from shelf s. Let A_s and P_s be the height and the profit of shelf s:

$$A_s := \max\{h_i : r_i^s > 0,\ i = 1, \ldots, m\} \quad \text{and} \quad P_s := \sum_{i=1}^{m} p_i r_i^s$$

2DKP - Gilmore and Gomory

By using integer variables y_s, $s \in \mathcal{S}$, the model reads:

$$
\begin{aligned}
\max \sum_{s \in \mathcal{S}} P_s y_s & \\
\sum_{s \in \mathcal{S}} r_i^s y_s \leq ub_i & \quad (i = 1, \ldots, m) \\
\sum_{s \in \mathcal{S}} A_s y_s \leq H & \\
y_s \in \mathbb{Z} & \quad (s \in \mathcal{S})
\end{aligned}
$$

By relaxing the integrality of the variables to $y_s \geq 0$ we get the *master* model. Dual constraints of the master read:

$$
P_s \leq \rho A_s + \sum_{i=1}^{m} \pi_i r_i^s \quad (\forall s \in \mathcal{S})
$$

2DKP - column generation

Columns with negative reduced cost can be detected by solving a BKP for each possible value of A_s, restricted to items with $h_i \leq A_s$:

$$
\begin{aligned}
Z(BKP) \ := \quad & \max \sum_{i=1}^{m} (p_i - \pi_i) r_i \\
\text{subject to} \quad & \sum_{i=1}^{m} w_i r_i \leq W \\
& 0 \leq r_i \leq ub_i \quad \text{integer} \quad (i = 1, \ldots, m)
\end{aligned}
$$

If there exists a feasible shelf s such that $Z(BKP) > \rho A_s$, then s is added to the master, otherwise the latter is optimally solved (one slave problem for each height).

2DKP - compact model (Lodi - Monaci, 2003)

For each item type i ($i = 1, \ldots, m$), define ub_i identical items j. Let $n = \sum_{i=1}^{m} ub_i$ be the overall number of items, ordered in such a way that $h_1 \geq h_2 \geq \cdots \geq h_n$. The model assumes that n potential shelves may be initialized: shelf k, if used, must be initialized by item k ($k = 1, \ldots, n$). We use the following binary variables:

$$x_{jk} = \begin{cases} 1 & \text{if item } j \text{ is cut from shelf } k \\ 0 & \text{otherwise} \end{cases} \qquad (k = 1, \ldots, n;\ j = k, \ldots, n)$$

For each variable x_{kk} ($k = 1, \ldots, n$), $x_{kk} = 1$ implies that shelf k is used and initialized by its corresponding item.

2DKP - compact model (M1 Lodi - Monaci, 2003)

The model is then as follows:

$$
\begin{aligned}
\text{M1} \qquad & \max \sum_{j=1}^{n} p_j \sum_{k=1}^{j} x_{jk} && \\
& \sum_{k=1}^{j} x_{jk} \leq 1 && (j = 1, \ldots, n) \\
& \sum_{j=k+1}^{n} w_j x_{jk} \leq (W - w_k)\, x_{kk} && (k = 1, \ldots, n-1) \\
& \sum_{k=1}^{n} h_k x_{kk} \leq H && \\
& x_{jk} \in \{0, 1\} && (k = 1, \ldots, n;\ j = k, \ldots, n)
\end{aligned}
$$

2DKP - compact model (M2 Lodi - Monaci, 2003)

Consider the items with the same shape's type together, but separate them with respect to the initialization of the shelves.
Any item of type i may be cut from shelves in $[1, \alpha_i]$, with $\alpha_i = \sum_{s=1}^{i} ub_s$ $(i = 1, \ldots, m)$ and $\alpha_0 = 0$.
Any shelf k can be used to cut items in $[\beta_k, m]$, with $\beta_k = \min\{r : 1 \leq r \leq m, \alpha_r \geq k\}$ $(k = 1, \ldots, n)$. Thus, β_k $(k = 1, \ldots, n)$ denotes the item type initializing shelf k.
Assuming $h_1 \geq h_2 \geq \cdots \geq h_m$, we have a first set of integer variables:

$$x_{ik} = \begin{cases} \text{number of items of type } i \text{ cut from shelf } k & \text{if } i \neq \beta_k \\ \text{number of } \textbf{additional} \text{ items of type } i \text{ cut from shelf } k & \text{if } i = \beta_k \end{cases}$$

where $i = 1, \ldots, m$; $k \in [1, \alpha_i]$.
A second set involves the following binary variables:

$$q_k = \begin{cases} 1 & \text{if shelf } k \text{ is used} \\ 0 & \text{otherwise} \end{cases} \qquad (k = 1, \ldots, n)$$

2DKP - compact model (M2 Lodi - Monaci)

The model is then as follows:

$$
\begin{aligned}
\text{M2} \quad & \max \sum_{i=1}^{m} p_i \Big(\sum_{k=1}^{\alpha_i} x_{ik} + \sum_{k=\alpha_{i-1}+1}^{\alpha_i} q_k \Big) \\
& \sum_{k=1}^{\alpha_i} x_{ik} + \sum_{k=\alpha_{i-1}+1}^{\alpha_i} q_k \le ub_i \qquad && (i = 1, \dots, m) \\
& \sum_{i=\beta_k}^{m} w_i x_{ik} \le (W - w_{\beta_k}) q_k \qquad && (k = 1, \dots, n) \\
& \sum_{k=1}^{n} h_{\beta_k} q_k \le H \\
& 0 \le x_{ik} \le ub_i \quad \text{integer} \qquad && (i = 1, \dots, m;\ k \in [1, \alpha_i]) \\
& q_k \in \{0, 1\} \qquad && (k = 1, \dots, n)
\end{aligned}
$$

2DKP - compact model (M2 Lodi - Monaci)

The LP relaxation of M2 is allowed to split an item into one parts, one initializing the shelf (q part) and some others which can be packed as "additional" parts (x parts).
Hence, the profit of the item is possibly taken into account completely, while the height of the shelf is only partially paid.

To avoid this drawback we add the following inequality:

$$\sum_{s=k}^{\alpha_i} x_{is} \leq ub_i - (k - \alpha_{i-1}) \quad (i = 1, \ldots, m;\ k \in [\alpha_{i-1} + 1, \alpha_i])$$

2dimensional 2staged Guillotine Bin Packing Problem (2DBPP)

We are given infinitely many identical stock rectangles with height H and width W, and a list of m rectangular shapes to be cut. Each shape's type i $(i = 1, \ldots, m)$ is characterized by a height h_i and a width w_i.
The problem calls for the determination of the 2staged guillotine cutting patterns needed to obtain all items by minimizing the number of used stock rectangles.

2DBPP - compact model (Lodi - Martello - Vigo, 2004)

Sort items so that $h_1 \geq h_2 \geq \cdots \geq h_n$. The model assumes that n potential levels are available, each associated with a different item i which initializes it, hence having the corresponding height h_i.

$$y_i = \begin{cases} 1 & \text{if item } i \text{ initializes level } i \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \ldots, n)$$

Only items j satisfying $j > i$ may be packed in level i (if this level is actually initialized by item i). Therefore the item packing is modeled by

$$x_{ij} = \begin{cases} 1 & \text{if item } j \text{ is packed into level } i \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \ldots, n-1;\ j > i)$$

2DBPP - compact model (Lodi - Martello - Vigo)

Similarly, we assume that n potential bins are available, each associated with a potential level k which initializes it.

$$q_k = \begin{cases} 1 & \text{if level } k \text{ initializes bin } k \\ 0 & \text{otherwise} \end{cases} \qquad (k = 1, \ldots, n)$$

Only levels i satisfying $i > k$ may be allocated to bin k (if this bin is actually initialized by level k). Therefore the level packing is modeled by

$$z_{ki} = \begin{cases} 1 & \text{if level } i \text{ is allocated to bin } k \\ 0 & \text{otherwise} \end{cases} \qquad (k = 1, \ldots, n-1; i > k)$$

2DBPP - compact model (Lodi - Martello - Vigo)

The ILP model follows:

$$\min \quad \sum_{k=1}^{n} q_k$$

$$\sum_{i=1}^{j-1} x_{ij} + y_j = 1 \qquad (j = 1, \dots, n)$$

$$\sum_{j=i+1}^{n} w_j x_{ij} \quad \leq (W - w_i) y_i \qquad (i = 1, \dots, n-1)$$

$$\sum_{k=1}^{i-1} z_{ki} + q_i \ = y_i \qquad (i = 1, \dots, n)$$

$$\sum_{i=k+1}^{n} h_i z_{ki} \quad \leq (H - h_k) q_k \qquad (k = 1, \dots, n-1)$$

2DBPP - compact model (Lodi - Martello - Vigo)

$$y_i \in \{0,1\} \qquad (i = 1, \dots, n)$$

$$x_{ij} \in \{0,1\} \qquad (i = 1, \dots, n-1;\ j > i)$$

$$q_k \in \{0,1\} \qquad (k = 1, \dots, n)$$

$$z_{ki} \in \{0,1\} \qquad (k = 1, \dots, n-1;\ i > k)$$

2DBPP - 2DCSP Set Covering Model

- Set covering model;
- Column generation;
- Branching.

The same considerations discussed for the one dimensional BPP and CSP apply to the 2dimensional case, with a suited definition of the variables.

Real world cutting applications

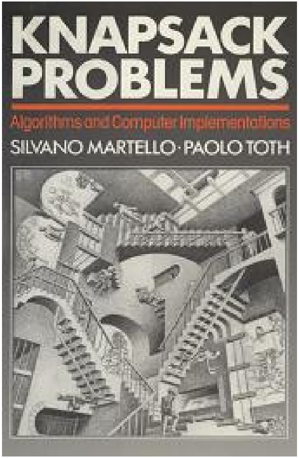
In some industries, e.g., wooden panel cutting, the cutting equipment productivity can be maximized by cutting several patterns in parallel.
Since the cutting pattern must be the same for all the boards, parallel cut limits the opportunities of stock usage minimization.
In addition to x_j, integer variables y_j denote the number of cutting cycles which are needed to obtain x_j patterns, namely, $y_j = \lceil x_j/\kappa \rceil$, where κ is the maximum number of patterns that can be cut in parallel.

$$\min \sum_{j=1}^{n} (c_j x_j + e_j y_j)$$

$$\sum_{j=1}^{n} p_j^i x_j \geq d_i \quad i = 1, \dots, m$$

$$y_j \geq \frac{x_j}{\kappa} \quad j = 1, \dots, n$$

$$x_j \in \mathbb{Z}^+, \qquad y_j \in \mathbb{Z}^+ \qquad j = 1, \dots, n$$



free download at http://www.or.deis.unibo.it/knapsack.html